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The integral representations of the q-Bessel-Macdonald functions

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Abstract

The q-Bessel-Macdonald functions of kinds 1, 2 and 3 are considered. Their representations by classical integral are constructed.

1 Introduction

The definitions of the q-Bessel-Macdonald functions (q-BMF) and their properties were given in [2]. Their representations by the Jackson q-integral have been constructed in [3, 4].

The double integral appears necessarily if we consider the problems of the harmonic analysis on the quantum Lobachevsky space. As the double Jackson q-integral is connected hardly with the q-lattice we have not any possibility to pass to the polar coordinates. So we are forced to introduce the usual double integral for q-functions.

In the first sections we remind the known formulas for q-Bessel functions and q-binomial.

2 The modified q-Bessel functions and the q-BMF

In [1] the q-Bessel functions were defined as follows:

$$J_\nu^{(1)}(z,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(z/2)^\nu \ {}_2\Phi_1(0,0;q^{\nu+1};q,-\frac{z^2}{4}), \tag{2.1}$$

$$J_\nu^{(2)}(z,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(z/2)^\nu \ {}_0\Phi_1(-;q^{\nu+1};q,-\frac{z^2q^{\nu+1}}{4}), \tag{2.2}$$

$$J_\nu^{(3)}(z,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(z/2)^\nu \ {}_1\Phi_1(0;q^{\nu+1};q,-\frac{z^2q^{\frac{\nu+1}{2}}}{4}). \tag{2.3}$$

where ${}_r\Phi_s$ is basic hypergeometric function [5],

$${}_r\Phi_s(a_1,\cdots,a_r;b_1,\cdots,b_s;q,z) = \sum_{n=0}^{\infty}\frac{(a_1,q)_n\dots(a_r,q)_n}{(q,q)_n(b_1,q)_n\dots(b_s,q)_n}[(-1)^nq^{n(n-1)/2}]^{1+s-r}z^n,$$

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$$(a,q)_n = \begin{cases} 1 & \text{for} \quad n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & \text{for} \quad n\geq 1, \end{cases}$$

$$(a,q)_\infty = \lim_{n\to\infty}(a,q)_n, \qquad (a_1,\dots,a_k,q)_\infty = (a_1,q)_\infty\dots(a_k,q)_\infty.$$

It allows to introduce the modified q-Bessel functions (q-MBFs) using (2.1), (2.2) and (2.3) similarly to the classical case [6].

Definition 2.1 *The modified q-Bessel functions are the functions*

$$I_\nu^{(j)}(z,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(z/2)^\nu \ {}_\delta\Phi_1\left(\underbrace{0,\dots,0}_{\delta};q^{\nu+1};q,\frac{z^2q^{\frac{\nu+1}{2}(2-\delta)}}{4}\right).$$

Here

$$\delta = \begin{cases} 2 & \text{for} \quad j=1 \\ 0 & \text{for} \quad j=2 \\ 1 & \text{for} \quad j=3. \end{cases} \tag{2.4}$$

Obviously,

$$I_\nu^{(j)}(z,q) = e^{-\frac{i\nu\pi}{2}}J_\nu^{(j)}(e^{i\pi/2}z,q), \qquad j=1,2,3.$$

In the sequel we consider the functions

$$I_\nu^{(1)}((1-q^2)z;q^2) = \sum_{k=0}^\infty \frac{(1-q^2)^k(z/2)^{\nu+2k}}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)}, \qquad |z|<\frac{2}{1-q^2}, \tag{2.5}$$

$$I_\nu^{(2)}((1-q^2)z;q^2) = \sum_{k=0}^\infty \frac{q^{2k(\nu+k)}(1-q^2)^k(z/2)^{\nu+2k}}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)}. \tag{2.6}$$

$$I_\nu^{(3)}((1-q^2)z;q^2) = \sum_{k=0}^\infty \frac{q^{k(\nu+k)}(1-q^2)^k(z/2)^{\nu+2k}}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)}, \tag{2.7}$$

where

$$\Gamma_{q^2}(\nu) = \frac{(q^2,q^2)_\infty}{(q^{2\nu},q^2)_\infty}(1-q^2)^{1-\nu}$$

is the q^2-gamma function. If $|q|<1$, the series (2.6) and (2.7) are absolutely convergent for all $z\neq 0$. Consequently, $I_\nu^{(2)}((1-q^2)z;q^2)$ and $I_\nu^{(3)}((1-q^2)z;q^2)$ are holomorphic functions outside a neighborhood of zero.

Remark 2.1

$$\lim_{q\to 1-0} I_\nu^{(j)}((1-q^2)z;q^2) = I_\nu(z), \qquad j=1,2,3.$$

Proposition 2.1 *The function $I_\nu^{(j)}((1-q^2)z;q^2)$ is a solution of the difference equation*

$$f(q^{-1}z)-(q^{-\nu}+q^\nu)f(z)+f(qz) = q^{-\delta}\frac{(1-q^2)^2}{4}z^2f(q^{1-\delta}z), \tag{2.8}$$

where $j=1,2,3$ are connected with $\delta=2,0,1$ by relations (2.4).

Corollary 2.1 *The function* $I^{(j)}_{-\nu}((1-q^2)z;q^2)$ *satisfies equation (2.8).*

Definition 2.2 *We define the q-Bessel-Macdonald function (q-BMF) for* $j=1,2,3$ *as follows [2, 4]:*

$$K^{(j)}_{\nu}((1-q^2)z;q^2)=$$
$$=\frac{1}{2}q^{-\nu^2+\nu}\Gamma_{q^2}(\nu)\Gamma_{q^2}(1-\nu)\left[A_{\nu}^{|1-\delta|}I^{(j)}_{-\nu}((1-q^2)z;q^2)-A_{-\nu}^{|1-\delta|}I^{(j)}_{\nu}((1-q^2)z;q^2)\right], \tag{2.9}$$

where

$$A_{\nu}=\sqrt{\frac{I^{(2)}_{\nu}(2;q^2)}{I^{(2)}_{-\nu}(2;q^2)}}. \tag{2.10}$$

As in the classical case, this definition must be extended to integral values of $\nu=n$ by passing to the limit in (2.9).

Definition 2.3 *The q-Wronskian of two solutions* $f^1_{\nu}(z)$ *and* $f^2_{\nu}(z)$ *of a second-order difference equation is defined as follows:*

$$W(f^1_{\nu},f^2_{\nu})(z)=f^1_{\nu}(z)f^2_{\nu}(qz)-f^1_{\nu}(qz)f^2_{\nu}(z).$$

If the q-Wronskian does not vanish, then any solution of the second-order difference equation can de written in form

$$f_{\nu}(z)=C_1f^1_{\nu}(z)+C_2f^2_{\nu}(z).$$

In this case the functions $f^1_{\nu}(z)$ and $f^2_{\nu}(z)$ form a fundamental system of the solutions of the given equation.

Proposition 2.2 *The functions* $I^{(j)}_{\nu}((1-q^2)z;q^2)$ *and* $K^{(j)}_{\nu}((1-q^2)z;q^2)$ *form a fundamental system of the solutions of equation (2.8)* ($z\neq\pm\frac{2q^{-r}}{1-q^2}, r=0,1,\dots$ *if* $j=1$).

This Proposition is following from

$$W(z)=\begin{cases}\frac{q^{-\nu}(1-q^2)}{2}A_{\nu}e_{q^2}(\frac{(1-q^2)^2}{4}z^2) & \text{for}\quad \delta=2\\ \frac{q^{-\nu}(1-q^2)}{2} & \text{for}\quad \delta=1\\ \frac{q^{-\nu}(1-q^2)}{2}A_{\nu}E_{q^2}(-\frac{(1-q^2)^2}{4}q^2z^2) & \text{for}\quad \delta=0.\end{cases}$$

(See [4].) Obviously, this function is defined for $z\neq\pm\frac{2q^{-r}}{1-q^2}, r=0,1,\dots$ if $j=1$ $(\delta=2)$ and does not vanish.

3 Some ancillary formulas

There is a q-analog of the classical binomial formula [5]

$$(1-z)^{-a}=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k,\ (a)_k=\frac{\Gamma(a+k)}{\Gamma(a)},\qquad |z|<1,$$

$$\frac{(q^{\alpha}z,q)_{\infty}}{(z,q)_{\infty}}=\sum_{k=0}^{\infty}\frac{(q^{\alpha},q)_k}{(q,q)_k}z^k,\qquad |z|<1.$$

We need in two generalizations of the q-binomial

$$r(a,b,z,q)=\frac{(az,q)_{\infty}}{(bz,q)_{\infty}} \tag{3.1}$$

$$R(a,b,\gamma,z,q^2)=\frac{(az^2,q^2)_{\infty}}{(bz^2,q^2)_{\infty}}z^{\gamma} \tag{3.2}$$

Proposition 3.1 *The function $R(a,b,\gamma,z,q^2)$ (3.2) satisfies the difference equation*

$$z^2[bq^{\gamma}R(a,b,\gamma,z,q^2)-aR(a,b,\gamma,qz,q^2)]=q^{\gamma}R(a,b,\gamma,z,q^2)-R(a,b,\gamma,qz,q^2). \tag{3.3}$$

The **Proof** see in [3]

It was shown in [3] that if $\alpha>\beta$ then

$$\frac{(-q^{2\alpha}z^2,q^2)_{\infty}}{(-q^{2\beta}z^2,q^2)_{\infty}}=\frac{(q^{2(\alpha-\beta)},q^2)_{\infty}}{(q^2,q^2)_{\infty}}\sum_{k=0}^{\infty}\frac{(q^{2(\beta-\alpha+1)},q^2)_k q^{2(\alpha-\beta-1)k}}{(q^2,q^2)_k(1+z^2q^{2\beta+2k})}. \tag{3.4}$$

Remark 3.1 *Let $a=\epsilon q^{2\alpha}, b=\epsilon q^{2\beta}, \epsilon=\pm 1$, in (3.2). Then if $q\to 1-0$ the difference equation (3.3) takes the form of the differential equation*

$$z(1-\epsilon z^2)R'(z)-[\gamma+\epsilon(2\alpha-2\beta-\gamma)z^2]R(z)=0 \tag{3.5}$$

with solution

$$R(z)=Cz^{\gamma}(1-\epsilon z^2)^{\beta-\alpha}.$$

Designate the q-derivative of $g(x)$ by

$$\partial_x g(x)=\frac{g(x)-g(qx)}{(1-q)x}.$$

It follows from

$$J_0^{(j)}((1-q^2)zs;q^2)=\sum_{k=0}^{\infty}\frac{(-1)^k q^{(2-\delta)k^2}(1-q^2)^{2k}(sz/2)^{2k}}{(q^2,q^2)_k^2} \tag{3.6}$$

that
$$\frac{2\partial_z}{1+q}J_0^{(j)}((1-q^2)q^{-1}zs;q^2)=-q^{1-\frac{\delta}{2}}sJ_1^{(j)}((1-q^2)q^{-\frac{\delta}{2}}zs;q^2), \tag{3.7}$$
$$\frac{2\partial_z}{1+q}J_0^{(j)}((1-q^2)zs;q^2)=-q^{1-\frac{\delta}{2}}sJ_1^{(j)}((1-q^2)q^{1-\frac{\delta}{2}}zs;q^2) \tag{3.8}$$
and
$$\frac{2\partial_s}{1+q}sJ_1^{(j)}((1-q^2)q^{1-\frac{\delta}{2}}zs;q^2)=q^{-\frac{\delta}{2}}zsJ_0^{(j)}((1-q^2)q^{1-\delta}zs;q^2) \tag{3.9}$$

Lemma 3.1 *If $F(x)$ is a differentiable (in classical sense) function in some neighborhood of zero then*
$$\lim_{\varepsilon\to 0}\int_{q\varepsilon}^{\varepsilon}\frac{F(x)}{x}dx=-F(0)\ln q. \tag{3.10}$$

Proof. Integrating by parts we have
$$\lim_{\varepsilon\to 0}\int_{q\varepsilon}^{\varepsilon}\frac{F(x)}{x}dx=\lim_{\varepsilon\to 0}\left[F(x)\ln x|_{q\varepsilon}^{\varepsilon}-\int_{q\varepsilon}^{\varepsilon}F'(x)\ln x dx\right]=$$
$$=\lim_{\varepsilon\to 0}\left[(F(\varepsilon)-F(q\varepsilon))\ln\varepsilon-F(q\varepsilon)\ln q-\int_{q\varepsilon}^{\varepsilon}F'(x)\ln x dx\right].$$
Using Lagrange's theorem and the theorem about mean value we obtain in the left side
$$\lim_{\varepsilon\to 0}\left[F'(\theta_1\varepsilon)(1-q)\varepsilon\ln\varepsilon-F(q\varepsilon)\ln q-(1-q)\varepsilon F'(\theta_2\varepsilon)\ln(\theta_2\varepsilon)\right],$$
where $\theta_1\in(0,1)$, $\theta_2\in(0,1)$. As $\lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon=0$ we have (3.10). ■

Lemma 3.2 *If $f(x)$ and $g(x)$ are integrable on $(0,\infty)$ and differentiable ones in zero (in classical sense) then the q-analog of formula of integration by parts for the q-derivative takes place*
$$\int_0^{\infty}\partial_x f(x)g(x)dx=f(0)g(0)\frac{\ln q}{1-q}-\int_0^{\infty}f(qx)\partial_x g(x)dx. \tag{3.11}$$

Proof.
$$\int_0^{\infty}\partial_x f(x)g(x)dx=\int_0^{\infty}\frac{f(x)-f(qx)}{(1-q)x}g(x)dx=$$
$$=\int_0^{\infty}\frac{f(x)g(x)-f(qx)g(qx)}{(1-q)x}dx-\int_0^{\infty}f(qx)\frac{g(x)-g(qx)}{(1-q)x}dx=$$
$$=\int_0^{\infty}\partial_x(f(x)g(x))dx-\int_0^{\infty}f(qx)\partial_x g(x)dx.$$
Using Lemma 3.1 calculate the first integral in the right side.
$$\int_0^{\infty}\partial_x(f(x)g(x))dx=\frac{1}{1-q}\lim_{\varepsilon\to 0}\left[\int_{\varepsilon}^{\infty}\frac{f(x)g(x)}{x}dx-\int_{\varepsilon}^{\infty}\frac{f(qx)g(qx)}{x}dx\right]=$$
$$=-\frac{1}{1-q}\lim_{\varepsilon\to 0}\int_{q\varepsilon}^{\varepsilon}\frac{f(x)g(x)}{x}dx=\frac{\ln q}{1-q}f(0)g(0).$$
The statement of the Lemma follows from here. ■

Corollary 3.1 *If $f(s)g(s)s^{-1}$ is integrable function on $(0,\infty)$ then*

$$\int_0^\infty f(s)\partial_s g(s)ds = -\int_0^\infty \partial_s f(s)g(qs)ds. \tag{3.12}$$

The **Proof** follows from $f(0)g(0)=0$ in this case. ∎

4 The integral representations of the q-BMFs

We will assume that z and s are the commuting variables.

Proposition 4.1 *q-BMF $K_\nu^{(j)}((1-q^2)z;q^2)$ for $\mathrm{Re}\nu>0$ can be represented as the integral*

$$K_\nu^{(j)}((1-q^2)z;q^2) = -\frac{q^{-\nu^2+\nu(1-\delta)}(1-q^2)}{2\ln q}\Gamma_{q^2}(\nu+1)A_\nu^{|1-\delta|}\times$$

$$\times (z/2)^{-\nu}\int_0^\infty \frac{(-q^{2\nu+2-\delta\nu}s^2,q^2)_\infty}{(-q^{-\delta\nu}s^2,q^2)_\infty} sJ_0^{(j)}((1-q^2)zs;q^2)ds, \tag{4.1}$$

where the constant A_ν is defined by (2.10) and $j=1,2,3$ are connected with $\delta=2,0,1$ by relations (2.4).

Proof. Consider the absolutely convergent integral

$$S^{(j)}(z) = \int_0^\infty f_\nu^{(j)}(s)sJ_0^{(j)}((1-q^2)zs;q^2)ds, \tag{4.2}$$

and require that $S^{(j)}(z)(z/2)^{-\nu}$ satisfies the difference equation (2.8). Then $S^{(j)}(z)$ satisfies the equation

$$S^{(j)}(q^{-1}z) - S^{(j)}(z) - q^{-2\nu}[S^{(j)}(z)-S^{(j)}(qz)] = q^{\nu(\delta-2)-\delta}(\frac{1-q^2}{2})^2 z^2 S^{(j)}(q^{1-\delta}z). \tag{4.3}$$

Substituting (4.2) in (4.3) and multiplying it on $\frac{2z^{-1}}{1-q^2}$ we obtain

$$\int_0^\infty f_\nu^{(j)}(s)s\frac{2z^{-1}}{1-q^2}[J_0^{(j)}((1-q^2)q^{-1}zs;q^2) - J_0^{(j)}((1-q^2)zs;q^2)]ds-$$

$$-q^{2\nu}\int_0^\infty f_\nu^{(j)}(s)s\frac{2z^{-1}}{1-q^2}[J_0^{(j)}((1-q^2)zs;q^2) - J_0^{(j)}((1-q^2)qzs;q^2)]ds =$$

$$= q^{\nu(\delta-2)-\delta}\frac{1-q^2}{2}z\int_0^\infty f_\nu^{(j)}(s)sJ_0^{(j)}((1-q^2)q^{1-\delta}zs;q^2)ds.$$

Due to (3.7) - (3.9) we can write

$$\int_0^\infty f_\nu^{(j)}(s)s^2J_1^{(j)}((1-q^2)q^{-\frac{\delta}{2}}zs;q^2)ds - q^{-2\nu+1}\int_0^\infty f_\nu^{(j)}(s)s^2J_1^{(j)}((1-q^2)q^{1-\frac{\delta}{2}}zs;q^2)ds =$$

$$= q^{\nu(\delta-2)-\frac{\delta}{2}} \frac{1-q^2}{2} s \int_0^\infty f_\nu^{(j)}(s) s J_0^{(j)}((1-q^2)q^{1-\delta} zs; q^2) ds$$

or

$$\int_0^\infty f_\nu^{(j)}(s) s^{2\nu+2} \frac{2\partial_s}{1+q} [s^{-2\nu+1} J_1^{(j)}((1-q^2)q^{-\frac{\delta}{2}} zs; q^2)] ds =$$

$$= -q^{\nu(\delta-2)} \int_0^\infty f_\nu^{(j)}(s) \frac{2\partial_s}{1+q} J_1^{(j)}((1-q^2)q^{-\frac{\delta}{2}} zs; q^2) ds.$$

Using (3.12) we obtain

$$\int_0^\infty \partial_s (f_\nu^{(j)}(s) s^{2\nu+2}) q^{-2\nu+1} s^{-2\nu+1} J_1^{(j)}((1-q^2)q^{1-\frac{\delta}{2}} zs; q^2) ds =$$

$$= -q^{\nu(\delta-2)} \int_0^\infty \partial_s f_\nu^{(j)}(s) qs J_1^{(j)}((1-q^2)q^{1-\frac{\delta}{2}} zs; q^2) ds.$$

Thus we come to the difference equation for $f_\nu^{(j)}(s)$

$$q^{-\delta\nu} s^2 [-f_\nu^{(j)}(s) + q^{2\nu+2} f_\nu^{(j)}(qs)] = f_\nu^{(j)}(s) - f_\nu^{(j)}(qs). \tag{4.4}$$

It follows from Proposition 3.1 the function

$$f_\nu^{(j)}(s) = \frac{(-q^{2\nu+2-\delta\nu} s^2, q^2)_\infty}{(-q^{-\delta\nu} s^2, q^2)_\infty} \tag{4.5}$$

satisfies to (4.4), and it follows from (3.4) integral (4.2) is absolutely convergent.

As $S^{(j)}(z)(z/2)^{-\nu}$ is a solution to (2.8) it can be represented as

$$S^{(j)}(z)(z/2)^{-\nu} = A I_\nu^{(j)}((1-q^2)z; q^2) + B K_\nu^{(j)}((1-q^2)z; q^2). \tag{4.6}$$

Let $j = 1$. As it follows from [2] $I_\nu^{(1)}((1-q^2)z; q^2)$ is a meromorphic function with the ordinary poles $z = \pm\frac{2q^{-r}}{1-q^2}$, $r = 0, 1, \ldots$, and $K_\nu^{(1)}((1-q^2)z; q^2)$ and the left side of (4.6) are the holomorphic functions in region $\mathrm{Re} z > 0$.

Let $j = 2, 3$. It is easily to show (see [4]) that $\lim_{z\to\infty} I_\nu^{(j)}((1-q^2)z; q^2) = \infty$, $\lim_{z\to\infty} K_\nu^{(j)}((1-q^2)z; q^2) = 0$ and the left side of (4.6) tends to zero if $z \to \infty$.

So for any $j = 1, 2, 3$ $A = 0$. Multiplying the both sides of (4.6) on $(z/2)^\nu$ and putting $z = 0$ we obtain from (2.9) and (2.5) - (2.7)

$$\int_0^\infty \frac{(-q^{2\nu+2-\delta\nu} s^2, q^2)_\infty}{(-q^{-\delta\nu} s^2, q^2)_\infty} s ds = \frac{B}{2} q^{-\nu^2+\nu} \Gamma_{q^2}(\nu) A_\nu^{|1-\delta|}. \tag{4.7}$$

Now calculate the integral in the left side of (4.7).

$$\int_0^\infty \frac{(-q^{2\nu+2-\delta\nu} s^2, q^2)_\infty}{(-q^{-\delta\nu} s^2, q^2)_\infty} s ds = \frac{q^{\delta\nu}}{2} \int_0^\infty \frac{(-q_1^{\nu+1} x, q_1)_\infty}{(-x, q_1)_\infty} dx = \frac{q^{\delta\nu}}{2} \int_0^\infty e_{q_1}(-x) E_{q_1}(q_1^{\nu+1} x) dx,$$

where $x = q^{-\delta\nu} s^2$ and $q_1 = q^2$ [5].

Note, that

$$\partial_x e_{q_1}(-x) = -\frac{1}{1-q_1} e_{q_1}(-x), \quad \partial_x E_{q_1}(q_1^{\nu+1}x) = \frac{q_1^{\nu+1}}{1-q_1} E_{q_1}(q_1^{\nu+2}x).$$

Using Lemma 3.2 we obtain

$$\int_0^\infty e_{q_1}(-x)E_{q_1}(q_1^{\nu+1}x)dx = -\ln q_1 + q_1^\nu \int_0^\infty e_{q_1}(-x)E_{q_1}(q_1^{\nu+1}x)dx.$$

So

$$\int_0^\infty e_{q_1}(-x)E_{q_1}(q_1^{\nu+1}x)dx = -\frac{\ln q_1}{1-q_1^\nu},$$

and we have

$$\int_0^\infty \frac{(-q^{2\nu+2-\delta\nu}s^2, q^2)_\infty}{(-q^{-\delta\nu}s^2, q^2)_\infty} sds = -\frac{q^{\delta\nu}\ln q}{1-q^{2\nu}}. \tag{4.8}$$

It follows from (4.8) that

$$B = -\frac{2q^{\nu^2-\nu(1-\delta)}\ln q}{(1-q^2)\Gamma_{q^2}(\nu+1)A_\nu^{|1-\delta|}},$$

and we have (4.1).■

Remark 4.1 *It follows from [2, Remark 5.1] $A_\nu \to 1$ if $q \to 1-0$, and it follows from Remark 3.1 that if $q \to 1-0$ we come to the classical integral representation of Bessel-Macdonald function [6]*

$$K_\nu(z) = \Gamma(\nu+1)\left(\frac{z}{2}\right)^{-\nu} \int_0^\infty (1+s^2)^{-\nu-1} sJ_0(zs)ds.$$

5 The representation of the q-BMFs by a double integral

We take $z, \ s \in \mathbf{C}$ in this section.

Consider function

$$\xi_\eta^{(\delta)}(s) = \sum_{n=0}^\infty \frac{q^{(2-\delta)\eta n^2}(1-q^2)^n}{(q^2,q^2)_n} s^n, \quad \eta \geq 0. \tag{5.1}$$

Obviously this series converges for any s if $(2-\delta)\eta > 0$ and for $|s| < \frac{1}{1-q^2}$ if $(2-\delta)\eta = 0$. Consider three cases.

1. $\eta = 0$. It follows from (5.1)

$$\xi_0^{(\delta)}(s) = e_{q^2}((1-q^2)s), \quad \delta = 2, 0, 1. \tag{5.2}$$

2. $\eta = 1$.

$$\xi_1^{(\delta)}(s) = \begin{cases} e_{q^2}((1-q^2)s) & \text{for} \quad \delta = 2 \\ E_{q^2}((1-q^2)qs) & \text{for} \quad \delta = 0 \\ {}_1\Phi_1(0; -q; q, -(1-q^2)q^{\frac{1}{2}}s) & \text{for} \quad \delta = 1. \end{cases} \tag{5.3}$$

3. $\eta=\frac{1}{2}$. It follows from (5.1)

$$\xi^{(\delta)}_{\frac{1}{2}}(s)=\begin{cases} e_{q^2}((1-q^2)s) & \text{for} \quad \delta=2 \\ {}_1\Phi_1(0;-q;q,-(1-q^2)q^{\frac{1}{2}}s) & \text{for} \quad \delta=0. \\ {}_3\Phi_3(0,0,0;-q^{\frac{1}{2}},iq^{\frac{1}{2}},-iq^{\frac{1}{2}};q^{\frac{1}{2}},-(1-q^2)q^{\frac{1}{4}}s) & \text{for} \quad \delta=1. \end{cases} \tag{5.4}$$

Assume $s=\rho e^{i\phi},\ z=re^{i\psi}$, and consider integral

$$J(r\rho)=\int_{-\pi}^{\pi}\xi^{(\delta)}_{\eta}(ir\rho e^{-i(\psi+\phi)})\xi^{(\delta)}_{1-\eta}(ir\rho e^{i(\psi+\phi)})d\phi. \tag{5.5}$$

Let $\delta<2,\ \eta=\frac{1}{2}$. In this case we can calculate this integral term by term.

$$J(r\rho)=\int_{-\pi}^{\pi}\sum_{n=0}^{\infty}\frac{q^{(2-\delta)\frac{n^2}{2}}(1-q^2)^n}{(q^2,q^2)_n}(ir\rho)^n e^{-in\phi}\sum_{m=0}^{\infty}\frac{q^{(2-\delta)\frac{m^2}{2}}(1-q^2)^m}{(q^2,q^2)_m}(ir\rho)^m e^{im\phi}d\phi=$$

$$=\sum_{n=0}^{\infty}\frac{q^{(2-\delta)\frac{n^2}{2}}(1-q^2)^n}{(q^2,q^2)_n}(ir\rho)^n\sum_{m=0}^{\infty}\frac{q^{(2-\delta)\frac{m^2}{2}}(1-q^2)^m}{(q^2,q^2)_m}(ir\rho)^m\int_0^{2\pi}e^{i(m-n)\phi}d\phi=$$

$$=2\pi\sum_{n=0}^{\infty}(-1)^n\frac{q^{(2-\delta)n^2}(1-q^2)^{2n}}{(q^2,q^2)^2_n}(r\rho)^{2n}.$$

It follows from (3.6) the last series is q^2-Bessel function and we have

$$J^{(j)}_0((1-q^2)2r\rho;q^2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\xi^{(\delta)}_{\frac{1}{2}}(ir\rho e^{-i(\psi+\phi)})\xi^{(\delta)}_{\frac{1}{2}}(ir\rho e^{i(\psi+\phi)})d\phi, \tag{5.6}$$

where $\xi^{(\delta)}_{\frac{1}{2}}$ is defined by (5.4) and $j=2,3$ are connected with $\delta=0,1$ by relations (2.4).

Let $\delta<2$ and $\eta=0$. Then we have the same result for $r\rho<\frac{1}{1-q^2}$, e.i.

$$J^{(j)}_0((1-q^2)2r\rho;q^2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e_{q^2}(i(1-q^2)r\rho e^{-i(\psi+\phi)})\xi^{(\delta}_1)(ir\rho e^{i(\psi+\phi)})d\phi, \tag{5.7}$$

where $\xi^{(de)}_1$ is defined by (5.3). The left side is a holomorphic function outside a neighborhood of zero, and so we can consider $J^{(j)}_0((1-q^2)2r\rho;q^2),\ j=2,3$ as the analytic continuation of (5.8).

Let $\delta=2$. It follows from (5.2) - (5.6) for $r\rho<\frac{1}{1-q^2}$

$$J^{(1)}_0((1-q^2)2r\rho;q^2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e_{q^2}(i(1-q^2)r\rho e^{-i(\psi+\phi)})e_{q^2}(i(1-q^2)r\rho e^{i(\psi+\phi)})d\phi. \tag{5.8}$$

In this case the both sides of (5.8) are the meromorphic functions with the ordinary poles in points $r\rho=\pm\frac{iq^{-k}}{1-q^2}$. So we will consider $J^{(1)}_0((1-q^2)2r\rho;q^2)$ as a analytic continuation of (5.8).

Now we can formulate

Proposition 5.1 *The q-BMF can be represented by double integral*

$$K\nu^{(j)}(2(1-q^2)|z|,q^2) = -\frac{q^{-\nu^2+\nu(1-\delta)}(1-q^2)}{8\pi\ln q}\Gamma_{q^2}(\nu+1)A_\nu^{|1-\delta|}|z|^{-\nu}\times \tag{5.9}$$

$$\times\int\int\frac{(-q^{2\nu+2-\delta\nu}\bar{s}s,q^2)_\infty}{(-q^{-\delta\nu}\bar{s}s,q^2)_\infty}e_{q^2}(i(1-q^2)\bar{z}\bar{s})\xi_1^{(\delta)}(izs)d\bar{s}ds$$

or

$$K\nu^{(j)}(2(1-q^2)|z|,q^2) = -\frac{q^{-\nu^2+\nu(1-\delta)}(1-q^2)}{8\pi\ln q}\Gamma_{q^2}(\nu+1)A_\nu^{|1-\delta|}|z|^{-\nu}\times \tag{5.10}$$

$$\times\int\int\frac{(-q^{2\nu+2-\delta\nu}\bar{s}s,q^2)_\infty}{(-q^{-\delta\nu}\bar{s}s,q^2)_\infty}\xi_{\frac{1}{2}}^{(\delta)}(i\bar{z}\bar{s})\xi_{\frac{1}{2}}^{(\delta)}(izs)d\bar{s}ds,$$

where $\xi_\eta^{(\delta)}$ *are defined by (5.2) - (5.4), the constant* A_ν *is defined by (2.10) and* $j=1,2,3$ *are connected with* $\delta=2,0,1$ *by relations (2.4).*

Proof. Substituting (5.6) - (5.8) in (4.1) we obtain (5.9) or (5.10) respectively. ■

Remark 5.1 *It ie easily to show that if* $q\to 1-0$ *the all functions (5.2) - (5.4) tend to the usual exponential* e^s*, and we come to the classical integral representation of Bessel-Macdonald function*

$$K_\nu(2\sqrt{\bar{z}z}) = \Gamma(\nu+1)(\sqrt{\bar{z}z})^{-\nu}\int\int(1+\bar{s}s)^{-\nu-1}\exp(i(\bar{z}\bar{s}+zs))d\bar{s}ds.$$

Bibliography

[1] Jackson F.H. "The application of basic numbers to Bessel's and Legendre's functions", Proc. London math. Soc. (2) **2** (1905) 192-220

[2] Olshanetskii M.A. and Rogov V.-B.K., "Modified q-Bessel functions and q-Macdonald functions", Sbornik: Math. **187**, 10 (1996) 1525-1544.

[3] Olshanetsky M. and Rogov V., "q-integral representations of modified q-Bessel functions and q-Macdonald functions", Sbornik: Math. **188**, 8 (1997) 1235-1258.

[4] Rogov V.-B.K. "q-Bessel-Macdonald functions" Preprint ITEP-TH-56/00, arXiv:math.QA/0010170.

[5] Gasper G. and Rahman M., Basic hypergeometric series, Cambridge Univ. Press, Cambridge 1990.

[6] Erdelyi A., Magnus W., Oberhattinger F.and Tricomi F.G., Higher transcendentalnfunctions, vol. 2 (based, in part, on notes left by Harry Bateman), McGraw-Hill, New York 1953.